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A Geometric Approach for Knot Selection in Convexity-Preserving Spline Approximation

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Abstract. A geometric approach is proposed for selecting the knots used in a parametric convexity-preserving B-spline approximation scheme. The approach automatically gives the necessary information about the shape suggested by the data which may be exact or not.

§1. Introduction

In many different fields such as medicine, physics, engineering and computer graphics the amount of data obtained through experimental and/or statistical surveys is very large. Consequently, the selection of a suitable small set of knots becomes an indispensable step within any efficient spline approximation scheme.

In this paper, large sets of exact and non-exact data points are approximated by means of a spline approximation scheme. So a knot selection strategy is necessary and this is provided by a geometric approach. In detail, the proposed approach is based on some weights suitably associated to the data points and directly computed from them. In addition, the method automatically defines the shape suggested by the data, here assumed planar and exact or not. Furthermore, the shape constraints for the approximating curve can be obtained in order to reproduce the desired behaviour.

Several other approaches have been studied, such as the knot removal methods [5,6], to reduce the number of parameters involved in an approximation problem. Interesting results are already available even for constrained approximation [1]. However, the approach introduced here differs from those because it does not reduce an initial large set of knots, but directly computes a suitable set. Furthermore, it does not require the starting approximation with many knots used in [1,5,6] for the weight definition (solving a minimization problem for each weight).

The proposed strategy has been tested on several examples for open and closed curves, and for exact and non-exact data points. The approximating

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curve is obtained by means of a convexity-preserving B-spline approximation scheme. The goodness of fit of the approximation has been estimated measuring the mean square distance of the data points from the resulting curve.

The outline of the paper is as follows. The problem and the method are presented in the next section. The strategies used to define the shape suggested by the data and to select the knots are introduced in Section 3, and they are given in more detail in the Appendix. Finally, in Section 4 the numerical results obtained for four sets of data points are given.

§2. The Problem and the Method

Let $\boldsymbol{P}_i \in \mathbb{R}^2, i = 0, \ldots, n$, be exact or non-exact data points, with n very large, and let $\mathcal{T} = \{t_i \in \mathbb{R}, i = 0, \ldots, n\}$ be an assigned set of associated strictly increasing parameter values. For non-exact data, let d be a positive assigned quantity so that $||\boldsymbol{P}_i - \boldsymbol{P}_i^e||_2 \leq d, i = 0, \ldots, n$, where $\boldsymbol{P}_i^e$ is the (unknown) exact data point corresponding to $\boldsymbol{P}_i$. We observe that, for simplicity's sake, the same maximum error value is assumed for all the data points.

It is well known that, if the data sets are very large and in particular the data are non-exact, the use of an approximation scheme is the only reasonable approach to construct a curve with the desired behaviour. Thus, here the aim is to first define the convexity constraints suggested by the data, and then to give a strategy for selecting a suitable small number of knots to construct a convexity-preserving least-square B-spline approximating curve.

Let $N_{jk}(t), j = 1-k, \ldots, nr-1$, be the usual B-splines of order k [4] defined with an extended knot vector $\Theta^* = \{\tau_{1-k}, \ldots, \tau_0, \ldots, \tau_{nr}, \ldots, \tau_{nr+k-1}\}$. Thus, we can introduce the B-spline representation of a spline curve

$$\boldsymbol{C}(t) = \sum_{j=1-k}^{nr-1} \boldsymbol{Q}_j N_{jk}(t), \qquad t \in [\tau_0, \tau_{nr}], \tag{1}$$

where $\boldsymbol{Q}_j, j = 1-k, \ldots, nr-1$ are de Boor control points.

The problem can be divided into three sub-problems. The definition of the shape suggested by the data (that is the determination of the convexity changes required to the approximating curve), the selection of the knots, and finally the construction of the convexity-preserving least-square B-spline approximating curve.

In particular, the shape suggested by the data is obtained through the procedure called "Shape Determination" (SD) described in the Appendix. SD uses some coefficients $u_i, i = 0, \ldots, n$, suitably associated with the data to establish in which parameter values zero curvature is required, and to determine the curvature sign in the interval $[\tau_0, \tau_{nr}]$. As the planar case is here considered, the curvature is defined as the function $\rho(t) = \frac{\dot{\boldsymbol{C}}(t) \times \ddot{\boldsymbol{C}}(t)}{||\dot{\boldsymbol{C}}(t)||_2^3}$, where $\boldsymbol{v} \times \boldsymbol{z} = v_1 \cdot z_2 - v_2 \cdot z_1, \ \forall \boldsymbol{v}, \boldsymbol{z} \in \mathbb{R}^2$). Thus, the procedure "Knots Selection" (KS) described in the Appendix selects the knot vector $\Theta = \{\tau_0, \ldots, \tau_{nr}\}$ by using the weights $w_i = |u_i|, i = 0, \ldots, n$. Θ^* is defined as the corresponding extended knot vector, taking into account whether the approximating curve is open or closed.

Finally, the last step is realized through the solution of a constrained parametric least-squares problem as a general constrained optimization problem. In fact, as the general parametric case is considered, the objective function $\sum_{i=0}^{n} ||\boldsymbol{P}_i - \boldsymbol{C}(t_i)||_2^2 = \sum_{i=0}^{n} ||\boldsymbol{P}_i - \sum_{j=1-k}^{nr-1} \boldsymbol{Q}_j N_{jk}(t_i)||_2^2$ is quadratic in the unknowns $\boldsymbol{Q}_j, j = 1-k, \ldots, nr-1$, but the convexity constraints are nonlinear.

§3. Data Shape Determination and Knot Selection Strategy

The SD procedure defines the shape suggested by the data, and KS selects the knots for constructing the B-spline curve. They are presented in the Appendix, but are commented upon here. Concerning the shape determination, SD computes the set $\mathcal{U} = \{u_0, \ldots, u_n\}$ whose sign variations are the curvature sign variations required for the approximating curve. If exact data are considered, $|u_i|$ is the reciprocal of the radius of the circle which passes through the points $\boldsymbol{P}_{i-1}, \boldsymbol{P}_i, \boldsymbol{P}_{i+1}$, and its sign is that of $\Delta_i = \frac{1}{2} det(\boldsymbol{L}_i, \boldsymbol{L}_{i+1})$, where $\boldsymbol{L}_i = \boldsymbol{P}_i - \boldsymbol{P}_{i-1}$. On the other hand, the sign of Δ_i is a reliable geometric information in the case of non-exact data only if the condition (2) of the theorem given in this section holds. Thus, the definition of $\mathcal{U}$ is suitably modified for non-exact data, using an input tolerance tol_d and considering the result given in the theorem below. In this case u_i is defined using the circle through the points $\boldsymbol{P}_{l_i}, \boldsymbol{P}_i$ and $\boldsymbol{P}_{r_i}$, where $\boldsymbol{P}_{l_i}$ and $\boldsymbol{P}_{r_i}$ are suitably selected points. In detail, $l_i \leq i-1$, $r_i \geq i+1, \sum_{k=l_i}^{i-1} ||\boldsymbol{P}_{k+1} - \boldsymbol{P}_k||_2 < tol_d \cdot Lp$ and $\sum_{k=i}^{r_i-1} ||\boldsymbol{P}_{k+1} - \boldsymbol{P}_k||_2 < tol_d \cdot Lp$, where Lp is the length of the polygonal joining the data points. SD computes the set $\Theta_S = \{\tau_0^s, \ldots, \tau_{ns}^s\} \subset \mathcal{T}$ and the set $\Sigma = \{\sigma_0, \ldots, \sigma_{ns-1}\}$, where Θ_S is such that $\tau_0^s = t_0, \tau_{ns}^s = t_n$ and zero curvature is required at each $\tau_i^s, i = 1, \ldots, ns-1$. The desired curvature sign between τ_i^s and τ_{i+1}^s is given by σ_i equal to -1 or 0 or 1. We observe that, for $d \neq 0$, in the procedure $u_i \neq 0$ is assumed to imply the existence of $k_i \in \{i-2, i-1, i\}$ such that $u_{k_i} \cdot u_i > 0, u_{k_i+1} \cdot u_i > 0$ and $u_{k_i+2} \cdot u_i > 0$. This hypothesis seems to be quite reasonable as n is considered very large.

Theorem 1. *Let $\boldsymbol{P}_i \in \mathbb{R}^2$ for $i = 0, \ldots, n$ be assigned non-exact data points, and let d be a small positive assigned quantity such that $||\boldsymbol{P}_i - \boldsymbol{P}_i^e||_2 \leq d, i = 0, \ldots, n$, where $\boldsymbol{P}_i^e$ is the (unknown) exact data point corresponding to $\boldsymbol{P}_i$. If $d < \frac{1}{2} \min_{i=1,\ldots,n} ||\boldsymbol{L}_i||_2$ with $\boldsymbol{L}_i = \boldsymbol{P}_i - \boldsymbol{P}_{i-1}$ and the condition*

$$\frac{||\boldsymbol{N}_i||_2}{||\boldsymbol{L}_i||_2 + ||\boldsymbol{L}_{i+1}||_2} > \frac{11}{4} d, \qquad i = 1, \ldots, n-1, \tag{2}$$

holds, then

$$\boldsymbol{N}_i^e \cdot \boldsymbol{N}_i > 0, \; i = 1, \ldots, n-1,$$

where $\boldsymbol{N}_i = \boldsymbol{L}_i \wedge \boldsymbol{L}_{i+1}$, $\boldsymbol{N}_i^e = \boldsymbol{L}_i^e \wedge \boldsymbol{L}_{i+1}^e$, $\boldsymbol{L}_i^e = \boldsymbol{P}_i^e - \boldsymbol{P}_{i-1}^e$ and the symbols " $\wedge$ " and "$\cdot$" denote the usual vector and scalar product, respectively.

Proof: We can write $\boldsymbol{P}_i^e = \boldsymbol{P}_i + \varepsilon_i \boldsymbol{v}_i$ with $0 \leq \varepsilon_i \leq d$ and $||\boldsymbol{v}_i||_2 = 1$. Then we have $\boldsymbol{L}_i^e = \boldsymbol{L}_i + \varepsilon_i \boldsymbol{v}_i - \varepsilon_{i-1} \boldsymbol{v}_{i-1}$ and $\boldsymbol{N}_i^e = \boldsymbol{N}_i + z_i$, where $z_i = \varepsilon_i \boldsymbol{v}_i \wedge \boldsymbol{L}_{i+1} -$

$\varepsilon_{i-1}\boldsymbol{v}_{i-1} \wedge \boldsymbol{L}_{i+1} + \varepsilon_{i+1}\boldsymbol{L}_i \wedge \boldsymbol{v}_{i+1} + \varepsilon_i\varepsilon_{i+1}\boldsymbol{v}_i \wedge \boldsymbol{v}_{i+1} - \varepsilon_{i-1}\varepsilon_{i+1}\boldsymbol{v}_{i-1} \wedge \boldsymbol{v}_{i+1} - \varepsilon_i\boldsymbol{L}_i \wedge \boldsymbol{v}_i + \varepsilon_{i-1}\varepsilon_i\boldsymbol{v}_{i-1} \wedge \boldsymbol{v}_i$. Thus, we can write $\boldsymbol{N}_i^e = (1 + f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}))\boldsymbol{N}_i$, where $f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}) = \frac{z_i \cdot \boldsymbol{N}_i}{||\boldsymbol{N}_i||^2}$. As a consequence, the assertion holds if $(1 + f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}))||\boldsymbol{N}_i||_2^2 > 0$, that is if $f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}) > -1$. Now with some algebra the following inequality can be easily obtained:

$$f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}) \geq -\frac{d}{||\boldsymbol{N}_i||_2}[2(||\boldsymbol{L}_i||_2 + ||\boldsymbol{L}_{i+1}||_2) + 3d]. \tag{3}$$

So, as $d < \frac{1}{2}\min_{i=1,\ldots,n} ||\boldsymbol{L}_i||_2$, $d < \frac{||\boldsymbol{L}_i||_2 + ||\boldsymbol{L}_{i+1}||_2}{4}$. Then, the inequality (3) and the hypothesis (2) imply that $f(\varepsilon_{i-1},\varepsilon_i,\varepsilon_{i+1}) \geq -\frac{d}{||\boldsymbol{N}_i||_2}(\frac{11}{4}(||\boldsymbol{L}_i||_2 + ||\boldsymbol{L}_{i+1}||_2)) > -1$, thus proving the theorem. □

Concerning the knot selection, in the KS procedure the knot vector $\Theta \subset \mathcal{T}$ is initialized with Θ_S. Then, for each interval $[\tau_{j-1}^s, \tau_j^s], j = 1,\ldots,ns$, it is checked if other parameters must be inserted in the knot vector using weights $w_i = |u_i|, i = 0,\ldots,n$. More precisely, a parameter value $t_l \in \mathcal{T} \cap (\tau_{j-1}^s, \tau_j^s)$ is inserted in Θ if one of the following conditions holds: either the corresponding weight w_l is big enough and $\boldsymbol{P}_l$ is far enough from all the other data related to the parameters previously introduced between τ_{j-1}^s and τ_j^s, or the corresponding weight w_l is not big enough but $\boldsymbol{P}_l$ is too far from them. For choosing reasonable values for the tolerances used in the previous consideration, denoted as tol_w, tol_{d1} and tol_{d2}, it is assumed that the distances are relative to an approximated curve length and the weights are relative to the maximum weight. The parameter values between τ_{j-1}^s and τ_j^s are ordered according to a decreasing order of the corresponding weights.

§4. Numerical Results

Four numerical tests are presented to analyze the performance of the approach. For all the considered tests the parameter values $t_i, i = 0,\ldots,n$, have been computed with the chord-length approach and a scaling such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. The approximated curve length L required as input by the KS procedure has been computed as the length of the piecewise linear interpolant of all the data points if $d = 0$ and of a suitably selected subset of them if $d > 0$.

A sequential quadratic programming method [2] is used to construct the approximating curve by means of the routine CONSTR of the Optimization toolbox of the Matlab package [3]. The set of control points used to start is chosen as the set of data points corresponding to the selected knots. Concerning the constraints, as $k = 4$ has been used in the experiments, four consecutive control points are required to generate a convex polygon if convexity is looked for in the corresponding curve segment. In addition three sequential collinear control points are required if zero curvature is asked at the corresponding knot.

For each test, a figure is given showing the corresponding approximating curve on the left, and the related curvature plot on the right. In all the figures the data points $\boldsymbol{P}_0, ..., \boldsymbol{P}_n$ are denoted with the symbol " $\cdot$ ", the points associated with the knots with the symbol " o ", and the points corresponding to the knots belonging to Θ_S with the symbol " $\otimes$ ".
The results are summarized in Table 1, where $n+1$ is the number of data points, tol_d is the input tolerance used in the procedure SD, tol_w, tol_{d1} and tol_{d2} are the input tolerances used in the procedure KS and $nr+1$ is the cardinality of the knot vector Θ. The symbol % denotes the percentage ratio $(nr+1)/(n+1)$. To estimate the goodness of fit of the approximation, the Mean Square Distance (MSD) of the data points $\boldsymbol{P}_i, i = 0, \ldots, n$, from the approximating curve is also given in the table.

	Test 1	**Test 2**	**Test 3**	**Test 4**
$n+1$	285	257	126	244
tol_d	-	0.2	0.2	0.5
tol_w	0.11	0.20	0.10	0.50
tol_{d1}	0.015	0.018	0.018	0.037
tol_{d2}	0.05	0.80	0.50	0.90
$nr+1$	36	30	22	8
%	12.6	11.7	17.5	3.3
MSD	1.08e-06	1.80e-03	2.19e-04	1.06e-01

Tab. 1. Results of the tests.

In the first test, data are considered exact ($d = 0$), while in the other tests they are non-exact.

Test 1 relates to a set of 285 exact data points that represent the alphabet capital letter " D ". In this case only a curvature sign variation to the approximating curve is required, as the curvature plot shows.

In Test 2, 257 non-exact data points are considered. They have been obtained by introducing a simulated random perturbation with $d = 0.4$ on the ordinates of the points $(x_i^e, y_i^e), i = 0, \ldots, 256$ defined as $x_i^e = -8 + dx \cdot i$, $y_i^e = 12\frac{sin(R_i)}{R_i}, i = 0, \ldots, 256$, where $R_i = sqrt(2(x_i^e)^2) + eps$, with eps denoting the round-off error and $dx = 0.0625$. We can observe that the simulated error does not preserve the symmetry of the data, and therefore the selected knots are not symmetric.

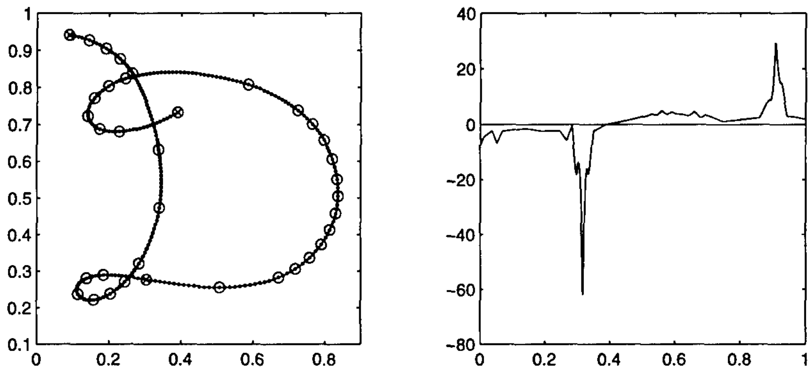

Fig. 1. Test 1: On the left the approximating curve, and on the right the related curvature plot.

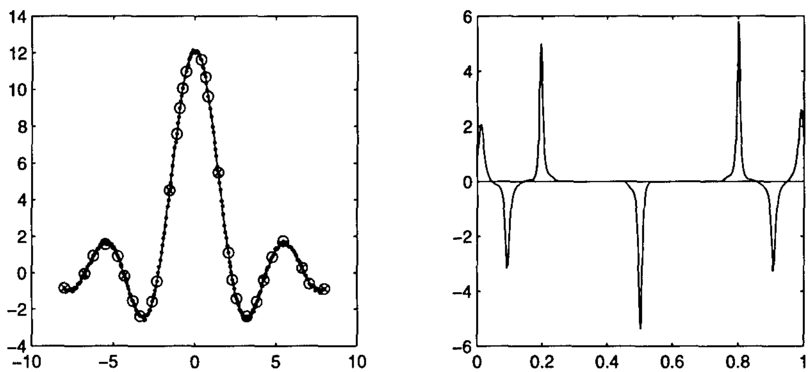

Fig. 2. Test 2: On the left the approximating curve, and on the right the related curvature plot.

In Test 3 the error in the 126 data is obtained by introducing a simulated random perturbation with $d = 0.12$ on both the coordinates of the points $(x_i^e, y_i^e), i = 0, \ldots, 125$ defined as $x_i^e = sin(4\pi \cdot dt \cdot i)$, $y_i^e = cos(2\pi \cdot dt \cdot i), i = 0, \ldots, 125$, where $dt = 0.008$. In this case, among the data there are sequences of quasi-collinear points, and we can observe that the approximating curve has corresponding almost straight line segments.

An application to an engineering problem is presented in Test 4. The 244 data points are derived from measurements effected in the Power Station located in Seraing (Belgium). The measurements are related to the active power of the alternator in the Central, observed on 03/01/1997 between 6:30:00 and 7:00:00. In this case the maximum value of the error is $d = 1$.

It should be noted that, the user needs to work in an interactive way for

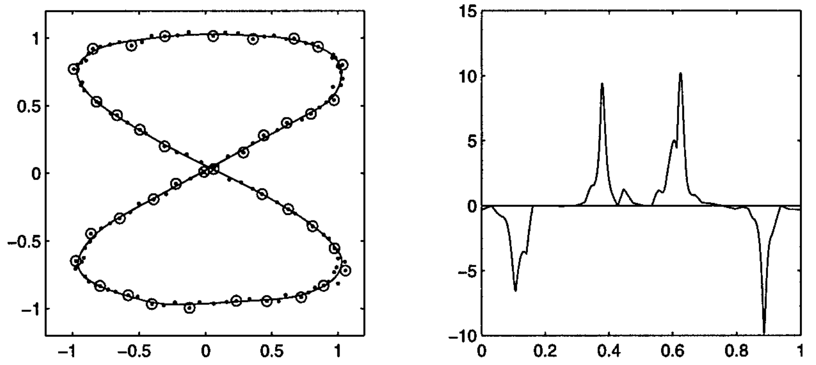

Fig. 3. Test 3: On the left the approximating curve, and on the right the related curvature plot.

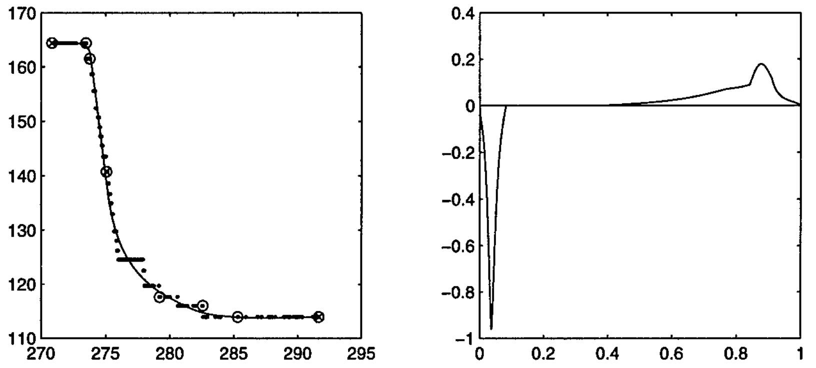

Fig. 4. Test 4: On the left the approximating curve, and on the right the related curvature plot.

selecting suitable values of the input tolerances. Obviously, a previous check of the data shape and of the data distribution is of great help.

§5. Appendix

SD Procedure (Shape Determination)

Input: $\mathcal{P} = \{\boldsymbol{P}_0, \ldots, \boldsymbol{P}_n\}$, d, tol_d, $\mathcal{T} = \{t_0, \ldots, t_n\}$
• Define two auxiliary suitable data points $\boldsymbol{P}_{-1}$ and $\boldsymbol{P}_{n+1}$
if $d = 0$
for $i = 0, \ldots, n$
$\bullet u_i = \frac{4\Delta_i}{||\boldsymbol{L}_i||_2 \cdot ||\boldsymbol{L}_{i+1}||_2 \cdot ||\boldsymbol{V}_i||_2}$

```
        with L_i = P_i − P_{i−1}, V_i = P_{i+1} − P_{i−1}, Δ_i = ½ det(L_i, L_{i+1})
    end
else
    •Lp = Σ_{k=1}^{n−1} ||P_{k+1} − P_k||_2
    for i = 0, ..., n
        •lev_{r_i} = max_{j=i+1,...,n+1} {j : Σ_{k=i}^{j−1} ||P_{k+1} − P_k||_2 < tol_d · Lp}
        •lev_{l_i} = min_{j=−1,...,i−1} {j : Σ_{k=j}^{i−1} ||P_{k+1} − P_k||_2 < tol_d · Lp}
    end
    for i = 0, ..., n
        • determine the biggest l_i ∈ {−1, ..., i − 1} and the
        smallest r_i ∈ {i + 1, ..., n + 1} such that (2) holds
        replacing P_{i+1} with P_{r_i} and P_{i−1} with P_{l_i}
        if l_i < lev_{l_i}  or  r_i > lev_{r_i}
            •u_i = 0
        else
            •u_i = 4Δ̃_i / (||L̃_i||_2 · ||L̃_{i+1}||_2 · ||Ṽ_i||_2)
            where L̃_i = P_i − P_{l_i}, L̃_{i+1} = P_{r_i} − P_i, Ṽ_i = P_{r_i} − P_{l_i},
            Δ̃_i = ½ det(L̃_i, L̃_{i+1})
        end
    end
end
•i = 1
•Θ_S = {t_0}
while i < n − 1
    if u_i = 0
        •Θ_S = Θ_S ∪ {t_{i−1}}
        • determine 0 ≤ id ≤ n − i − 1 such that
            u_i = ... = u_{i+id} = 0 and
            u_{i+id+1} ≠ 0 or id = n − i − 1
        •Θ_S = Θ_S ∪ {t_{⌊(2i+id)/2⌋}}
        •Θ_S = Θ_S ∪ {t_{i+id+1}}
        •i = i + id + 2
    elseif u_i · u_{i+1} < 0
        if |u_i| < |u_{i+1}|
            •Θ_S = Θ_S ∪ {t_i}
            •i = i + 1
        else
            •Θ_S = Θ_S ∪ {t_{i+1}}
            •i = i + 2
        end
    else
        •i = i + 1
    end
end
•Θ_S = Θ_S ∪ {t_n} =def {τ_0^s, ..., τ_{ns}^s}
```

for $j = 0, \ldots, ns$
    $\bullet s(j) = i$ where $\tau_j^s = t_i$
end
for $j = 0, \ldots, ns - 1$
    if $s(j+1) = s(j) + 1$
        $\bullet \sigma_j = 0$
    else
        $\bullet \sigma_j = sgn(u_{s(j)+1})$
    end
end
Output: $\mathcal{U} = \{u_0, \ldots, u_n\}$, $\Theta_S = \{\tau_0^s, \ldots, \tau_{ns}^s\}$, $\Sigma = \{\sigma_0, \ldots, \sigma_{ns-1}\}$.

KS Procedure (Knot Selection)

Input:
    $\mathcal{P} = \{\boldsymbol{P}_0, \ldots, \boldsymbol{P}_n\}, \mathcal{T} = \{t_0, \ldots, t_n\}$,
    $\mathcal{U} = \{u_0, \ldots, u_n\}$, $\Theta_S = \{\tau_0^s, \ldots, \tau_{ns}^s\}$,
    L, tol_w, tol_{d1}, tol_{d2} $(tol_{d1} << tol_{d2})$
$\bullet w_i = |u_i|$, $i = 0, \ldots, n$
$\bullet \Theta = \Theta_S$
$\bullet w_{max} = max\{w_0, \ldots, w_n\}$
for $j = 0, \ldots, ns$
    $\bullet s(j) = i$ where $\tau_j^s = t_i$
end
for $j = 1, \ldots, ns$
    $\bullet s_j = s(j) - s(j-1) + 1$
    $\bullet$ let $\{i_1, \ldots, i_{s_j}\}$ be the index permutation of $\{s(j-1), \ldots, s(j)\}$ such that the weights $w_{i_1}, \ldots, w_{i_{s_j}}$ are in decreasing order
    $\bullet \Theta_j = \{t_{s(j-1)}, t_{s(j)}\}$
    $\bullet \mathcal{P}_{R_j} = \{\boldsymbol{P}_{s(j-1)}, \boldsymbol{P}_{s(j)}\}$
    for $k = 1, \ldots, s_j$
        if $\frac{w_{i_k}}{w_{max}} > tol_w$ and $\forall \boldsymbol{P}_r \in \mathcal{P}_{R_j}$ $\frac{\|\boldsymbol{P}_{i_k} - \boldsymbol{P}_r\|_2}{L} > tol_{d1}$
            $\bullet \Theta_j = \Theta_j \cup \{t_{i_k}\}$
            $\bullet \mathcal{P}_{R_j} = \mathcal{P}_{R_j} \cup \{\boldsymbol{P}_{i_k}\}$
        elseif $\frac{w_{i_k}}{w_{max}} \le tol_w$ and $\forall \boldsymbol{P}_r \in \mathcal{P}_{R_j}$ $\frac{\|\boldsymbol{P}_{i_k} - \boldsymbol{P}_r\|_2}{L} > tol_{d2}$
            $\bullet \Theta_j = \Theta_j \cup \{t_{i_k}\}$
            $\bullet \mathcal{P}_{R_j} = \mathcal{P}_{R_j} \cup \{\boldsymbol{P}_{i_k}\}$
        end
    end
    $\bullet \Theta = \Theta \cup \Theta_j$
end
Output: $\Theta = \{\tau_0, \ldots, \tau_{nr}\}$

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